Resummation of jet mass with and without a jet veto

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Calculating the distribution of jet masses in high-energy collisions is challenging because fixed-order perturbation theory breaks down near the peak region, and because multiple scales complicate the resummation. To avoid using a jet veto, one can consider inclusive observables, in which every particle is in a jet. We demonstrate that calculating the mass of the hardest jet in multijet events can be problematic, and we give an example of an inclusive observable, asymmetric thrust, which can be resummed to next-to-next-to-leading logarithmic accuracy. Exclusive observables with out-of-jet regions are more complicated. Even for e^+e^- dijet events at energy Q, to calculate the mass m of jets of size R, one must impose a veto on the energy ω of extra jets to force dijet kinematics; then there are both $\log m/Q$ and $\log m/\omega$ singularities. To proceed, we suggest a refactorization of the soft function in the small R limit To justify this refactorization, we show that the expansion of the resummed distribution is in excellent agreement with fixed order. This motivates considering the expansion around small R as a useful handle on producing phenomenologically useful resummed jet mass distributions. The strong evidence we give for refactorization at small R is independent of non-global logarithms, which are not the subject of this paper.

1. INTRODUCTION

The Large Hadron Collider has already produced enormous numbers of high energy jets. These jets provide a wealth of information about QCD as well as a vital area for exploration of new physics. While much activity has been devoted to the calculation of the distribution of jets, for example using leading order (LO) or next-to-leading order (NLO) perturbative calculations in QCD, very little has been said with any precision about the substructure of the jets themselves. This is unfortunate, because the substructure of jets may be critical to finding new physics. In fact, there has been much progress over the last few years on practical methods using jet substructure to separate signal from background at the LHC [1–3]. However, even for the simplest of jet shapes, the jet mass, these studies are forced to rely on the Monte Carlo approximation. Needless to say, it would be great to have a systematically improvable way to calculate jet shapes, and it may even prove essential if new physics shows up in one of these channels.

The main theoretical difficulty with calculating substructure is that fixed-order perturbation theory is in general a very bad approximation. For example, at LO, jet mass distributions for small mass diverge as $d\sigma \sim 1/m^2$, while the measured distributions have a peak at small mass and then go to zero. The physical peak structure is easily explained qualitatively as being due to large logarithmic terms $\alpha_s^i \log^j(m^2/E^2)$, which dominate over the fixed-order expansion in α_s for small enough mass. The leading large logarithms (LL) are resummed with the Monte Carlos (or analytically [4]); however, until recently, it was impossible to go beyond this order. The difficulty stems from the fact that while the leading Sudakov double logarithms are universal, subleading logarithms are not. The coefficients of the subleading logs

depend on many variables besides the mass, including the jet size, the jet algorithm, and the directions and energies of the other jets. Sorting out all these effects and all the scales is a daunting task. However, with recent advances in effective field theory, it is now possible to make systematically improvable calculations of jet substructure, at least in some kinematical regimes.

While we are mainly interested in jet masses at hadron colliders, from a theoretical perspective it is instructive to start with e^+e^- machines. In e^+e^- collisions, jet shapes can be approached by first studying hemisphere jets and inclusive event shapes like thrust. The factorization theorem in Soft-Collinear Effective Theory (SCET) writes the doubly differential hemisphere mass distribution as a convolution of a hard function, describing the short distance $e^+e^- \to q\bar{q}$ process, jet functions, describing the collinear degrees of freedom in the hemisphere jets, and a soft function, which describes radiation which can connect the jets to each other [5, 6]:

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dm_L^2 dm_R^2} = H(Q^2,\mu)\int dk_L dk_R$$
$$\times J(m_L^2 - k_L Q)J(m_R^2 - k_R Q)S_{\text{hemi}}(k_L, k_R, \mu), \quad (1)$$

Here m_L (m_R) is the invariant mass of the 4-vector sum of the momenta of particles in the left (right) hemisphere, with respect to the thrust axis, and σ_0 is the born cross section for e^+e^- to dijets. The challenge which we address in this paper is to modify this result to describe jets of arbitrary size R while still resumming all of the large logarithms.

Before considering non-hemisphere jets, a few comments are in order about non-global logarithms (NGL). Non-global logarithms [7] arise in situations where there are multiple scales. They become relevant, for example, when one integrates the double differential mass distri-

bution over m_R^2 to produce the left-hemisphere mass distribution. The full QCD calculation of $d\sigma/dm_L^2$ involves integrals over hard emissions into the right side which are not in the soft or collinear sectors present in the leading SCET Lagrangian. To account for these non-global logs in SCET would require incorporating 3-jet and higher order operators in SCET (see [8, 9]). These logs appear very difficult to resum since there is no known simplification for multiple hard emissions in QCD. However, they can be avoided by carefully chosen observables. For example, thrust and heavy jet mass do not have the non-global problems that left-hemisphere or light-jet mass have.

There are other types of non-global logs that are relevant even for an observable which is well described by the leading operator in SCET. For example, consider the doubly differential jet mass distribution in the regime $m_L \ll m_R \ll Q$. In this regime, the above factorization formula is valid because hemisphere masses (m_L and m_R) are small. This is in contrast to the left-jet mass, where in integrating the right-jet mass to its kinematic limit one enters the regime with $m_R \sim Q$ where the factorization formula fails. So when $m_L, m_R \ll Q$, the exact distribution is completely dominated by the IR degrees of freedom included in SCET. However, SCET *does not* guarantee that the logarithms of m_L and m_R can be resummed using renormalization group evolution (RGE). In particular, logs of the form $\log(m_L^2/m_R^2)$ are not completely fixed by the RGE. If by brute force one could calculate to all orders in SCET, the result would reproduce the m_L and m_R dependence of full QCD when both masses are small [33]. In this paper, we do not attempt to predict or resum these challenging NGLs. Instead, we give evidence that a certain soft function refactorizes in the small R limit. This provides an orthogonal direction in the study of jet shapes to the investigation of NGLs.

Now consider non-hemisphere jets. There has already been some work in SCET on understanding finite jet-size effects. For example, jet functions for the k_T and cone jets have been calculated at NLO [12, 13] , agreement with QCD calculations of the 2-jet limit has been shown for small R [14], and RG invariance at the 1-loop level has been demonstrated for jet mass and jet angularity distributions [13, 15]. These works have provided a number of non-trivial consistency checks on the SCET formalism. However, it has not yet been demonstrated that any of the jet shapes exponentiate. Due the presence of multiple scales, such as the jet size R and the amount of in-jet and out-of-jet radiation, there is plenty of opportunity for logs like $\log(m_L^2/m_R^2)$ in the hemisphere case, which are not fixed by RGE, to inhibit resummation.

In this paper, we will consider various jet mass observables. We focus on $e^+e^- \to$ dijets events, since analytical calculations can be performed to order α_s and numerical calculations to α_s^2. To lend concreteness to the discussion, all events are to be clustered using the Cambridge/Aachem (CA) algorithm (see [16]). That is, take the pair of particles with the smallest angle and merge them if the distance

$$R_{ij} = \frac{1}{2}(1 - \cos\theta_{ij}) \qquad (2)$$

is less than R. If no particles have $R_{ij} < R$ then stop. The jets can be ordered by energy, and E_i, p_i^μ will denote the energy and 4-momentum of the i-th most energetic jet. This parameter R can be translated into other jet size measures, such as ones used at hadron colliders.

We begin by considering observables which avoid a jet veto. To do this, one has to have an inclusive observable whose singular region forces the threshold in which both jet masses are small. The example we consider is asymmetric jet masses and asymmetric thrust, which are defined in Section 2. In this case, we show that the asymmetric jet mass of the hardest jet does not agree with any calculation in SCET, while the asymmetric jet mass of the quark (or anti-quark) jet cannot be defined in QCD. When one averages over both jets, the results do agree, and we check at order α_s^2 that the next-to-next-to-leading logarithms are being resummed.

Next, we move on to observables which have an out-of jet region. To resum the large logarithms of jet mass in this case, one must reject events in which there is hard radiation in the out-of-jet region. To do this, we introduce a parameter ω which controls the out-of-jet energy. Resumming logs of jet mass in this case is trickier than for asymmetric thrust. We argue that expanding around a different threshold $R \sim 0$, provides a new handle on these complicated multiscale observables. We postulate a refactorized form of the soft function, valid at small R, and then show it gives much improved agreement with the NLO predictions of QCD, than using SCET alone. This indicates than an expansion around small R maybe be a promising direction for precision jet substructure calculations.

2. INCLUSIVE JET MASS: ASYMMETRIC THRUST

In this section, we consider a simple inclusive jet mass observable, introduced in [17]. Cluster the particles into jets of size R using Cambridge/Aachen (see above). Then asymmetric thrust τ_A is defined as the sum of the mass-squared of one primary jet and the mass-squared of everything else in the event. Which jet is chosen to be primary will be discussed shortly.

There are three main reasons for considering this odd-sounding observable: First, every particle contributes to the value of τ_A. So the region with $\tau_A \ll 1$ forces dijet kinematics and the SCET factorization formula should hold. Second, there is R-dependence in the jet mass so one can study R-dependence of threshold logs. This provides a warm up to more complicated exclusive jet mass observables. Third, asymmetric thrust may be useful to measure jet masses at hadronic colliders. Indeed, it was suggested first in the context of dijet events in pp collisions. But more simply, in direct-photon[18] or W/Z [19]

events at high p_T, asymmetric thrust would measure the same singular behavior of the jet mass as any other jet mass measure, but it may be easier to compute and is likely to undergo dynamic threshold enhancement. The hadron collider applications are not the subject of this paper.

In dijet events at e^+e^- machines, the first question is which jet should be the primary jet for asymmetric thrust. In data (or in full QCD), the obvious candidate is the hardest jet. Let us call this τ_{A_1}. That is,

$$\tau_{A_1} = \frac{1}{Q^2}(m_1^2 + m_{\bar{1}}^2) \tag{3}$$

where m_1 is mass of the hardest jet, (largest energy), and $m_{\bar{1}}$ is the mass of all the particles not clustered into the hardest jet. A quick study of the kinematics shows that the τ_{A_1} distribution in QCD at order α_s is exactly equal to the $\tau = 1 - T$ distribution, with no R-dependence. That is,

$$\frac{1}{\sigma_0}\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_{A_1}}\right]_{\mathrm{QCD}} = \delta(\tau_{A_1}) + \frac{C_F\alpha_s}{4\pi}\left\{\left[\frac{-8\ln\tau_{A_1} - 6}{\tau_{A_1}}\right]_+ \right.$$
$$\left. + \left[-2 + \frac{2\pi^2}{3}\right]\delta(\tau_{A_1}) + \cdots\right\} \tag{4}$$

where the $\cdots$ are regular in τ_{A_1}.

Now we turn to the calculation of asymmetric thrust in SCET. Note that $\tau_A \to 0$ forces dijet kinematics independent of which jet is chosen to be the primary one. Thus, the effective theory should agree with QCD in the singular region for any R. In fact, if we are interested in the limit that $\tau_A \to 0$, then the size of the jet is irrelevant to the singularities associated with collinear radiation. For this reason, we can use the inclusive jet function. Indeed, the use of the inclusive jet function for the hemisphere mass case has to be justified using the same logic. In contrast, for the soft function, the R-dependence is important. The calculation of the soft function is problematic for τ_{A_1}. The problem can be traced to the fact that the soft function in SCET has no access to the energy of the jets. In the region of phase space in which the gluon is soft but not within R of either quark, the most energetic jet is not well defined. In QCD, there is no such ambiguity. We conclude that SCET cannot calculate τ_{A_1}.

Instead, one might consider calculating τ_{A_q}, where the primary jet is taken to be the quark. When the radiation is clustered with the quark, we find that the contribution to the soft function from radiation inside a cone of size R is

$$S_R^{\mathrm{in}}(k,\mu) = \delta(k) + \frac{C_F\alpha_s}{4\pi}\left(-2\ln^2\frac{R}{1-R} + \frac{\pi^2}{3}\right)\delta(k)$$
$$+ \frac{C_F\alpha_s}{4\pi}\left[\frac{-16\ln\frac{k}{\mu} + 8\ln\frac{R}{1-R}}{k}\right]_\star^{[k,\mu]} \tag{5}$$

where the $\star$-distribution notation can be found in [6]. When the radiation is outside the quark's cone, the contribution from the complimentary region (everything but the cone) is simply $S_{1-R}^{\mathrm{in}}(k,\mu)$. When we convolute these two contributions to the soft function with the inclusive (R-independent) jet functions, as in Eq. (1), we produce the τ_{A_q} distribution in SCET. The result is

$$\frac{1}{\sigma_0}\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_{A_q}}\right]_{\mathrm{SCET}} = \delta(\tau_{A_q}) + \frac{C_F\alpha_s}{4\pi}\left\{\left[\frac{-8\ln\tau_{A_q} - 6}{\tau_{A_q}}\right]_+ \right.$$
$$\left. + \left[-2 + \frac{2\pi^2}{3} - 4\ln^2\frac{R}{1-R}\right]\delta(\tau_{A_q}) + \cdots\right\} \tag{6}$$

Comparing to Eq. (4) we see that the asymmetric thrust distribution in SCET has R dependence, while the QCD result for τ_{A_1} does not, so the distributions do not agree. Of course, it is not unreasonable that they do not agree, since they are different observables.

If we repeat the QCD calculation, but always defining the primary jet as the quark jet, we find a result which agrees exactly with the SCET prediction, Eq. (6), up to non-singular terms. Unfortunately, this quark-jet asymmetric thrust is not infrared safe, and therefore cannot be used to measure jet mass. Indeed, testing τ_{A_q} at next-to-leading order by taking the primary jet to be the hardest one containing a quark parton, we find negative cross sections.

Instead of attempting to define a quark jet in QCD (see [20] for an example of how this might be done), we can get agreement by averaging the jets. Note first that the hardest jet asymmetric thrust, which we calculated in QCD, is not the average of the quark and anti-quark asymmetric thrust in SCET. However, if we calculate asymmetric thrust in QCD with the second hardest jet as primary, we find

$$\frac{1}{\sigma_0}\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_{A_2}}\right]_{\mathrm{QCD}} = \delta(\tau_{A_2}) + \frac{C_F\alpha_s}{4\pi}\left\{\left[\frac{-8\ln\tau_{A_2} - 6}{\tau_{A_2}}\right]_+ \right.$$
$$\left. + \left[-2 + \frac{2\pi^2}{3} - 8\ln^2\frac{R}{1-R}\right]\delta(\tau_{A_2}) + \cdots\right\} \tag{7}$$

Thus the average of the harder and softer jet asymmetric thrust distributions in QCD is the same as the average of the quark and anti-quark distributions in SCET at order α_s. While this may not be surprising, it illustrates the importance of being careful not to discuss quark jet and gluon-jet masses in QCD without a proper infrared safe observable definition [21, 22].

Now that the observables agree at order α_s we can check resummation by expanding the resummed results in SCET to order α_s^2 and comparing against the output of the numerical QCD calculation using the program EVENT 2 [4]. We look at the observable

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_A} = \frac{\mathrm{d}\sigma}{\mathrm{d}\tau_{A_1}} + \frac{\mathrm{d}\sigma}{\mathrm{d}\tau_{A_2}} \tag{8}$$
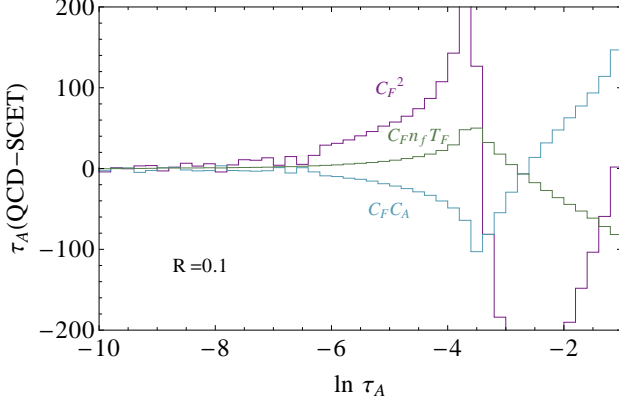
FIG. 1: Difference between QCD and SCET for the sum $\tau_{A_1}\frac{d\sigma}{d\tau_{A_1}} + \tau_{A_2}\frac{d\sigma}{d\tau_{A_2}}$ at order α_s^2. That the difference vanishes at very small τ_A (left side), shows that SCET reproduces all of the large logs of this distribution up to NNLL.

and compare to the result of adding the quark and anti-quark jet asymmetric thrust distributions in SCET.

Figure 1 shows the difference between the QCD and SCET predictions for τ_A as a function of $\ln \tau_A$ in the singular region, separated by color structure. Figure 2 shows the distribution for a number of different values of R. In all cases, there is excellent convergence at $\tau_A \to 0$, showing that SCET accounts for all the singularities at NLO. This demonstrates that SCET can resum all of the large logarithms of τ_A for any R. At very small R, these may not be the dominant part of the distribution, due to large $\ln R$ terms, but we have not attempted to resum logs of R for τ_A.

As observed in [17], τ_A may be a promising to measure the jet mass in $W/Z/\gamma$+jet events at hadron colliders. It can be calculated at the machine threshold and should undergo dynamical threshold enhancement making it relevant at the partonic threshold as well. An alternative to invoking dynamical threshold enhancement would be to have a jet mass observable which is not sensitive to the whole event. For example, one could look at the mass of a jet in exclusively 2-jet events. To do this requires a veto on additional radiation. So we now turn to the calculation of jet mass with a jet veto in SCET.

3. EXCLUSIVE JET MASS: DIJETS WITH A VETO

In this section, we will consider the exclusive case, where jets are clustered using Cambridge/Aachen, as before, but only the two hardest jets are considered. For asymmetric thrust, there was one primary jet, with the secondary jet containing all the other radiation in the event. For an exclusive dijet sample, we have to veto additional jets. We do this by imposing an energy cutoff ω on the 3rd most energetic jet: $E_3 < \omega$ (at tree level and $\mathcal{O}(\alpha_s)$ this is equivalent to vetoing on the total out-of-
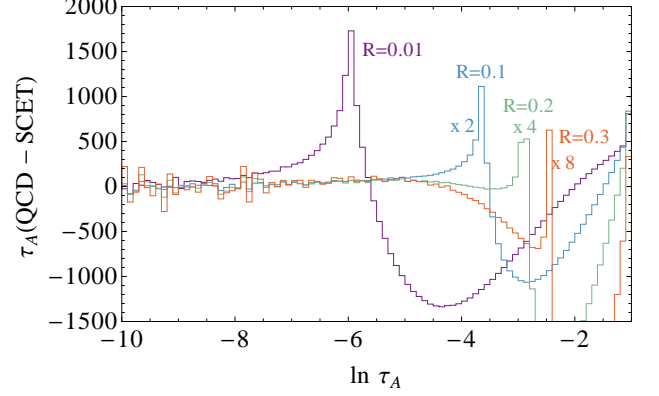

FIG. 2: Difference between QCD and SCET for $\tau_{A_1}\frac{d\sigma}{d\tau_{A_1}}$ at order α_s^2 for a variety of jet sizes.

jet radiation, up to a contribution in which the jet mass is exactly zero). We will consider several different dijet mass observables and explore to what extent SCET can reproduce the QCD results and perform resummation of the large logarithms.

The first observable we will consider is the doubly differential cross section in the masses m_1 and m_2 of the two most energetic jets. As we saw with asymmetric thrust, one cannot order the jets by energy in SCET since, after matching, the operator only has access to the jets' energies through the labels, each of which has the same magnitude. Instead in SCET one can calculate the quark and anti-quark jet mass, but these are not infrared safe in QCD. As with asymmetric thrust, we will see that a meaningful comparison can be made between an average over the two jets or more simply the sum of the jet masses squared.

First consider the QCD calculation of $d^2\sigma/dm_1^2 dm_2^2$. The tree level process $e^+e^- \to q\bar{q}$ and its one-loop virtual corrections contribute to the $\delta(m_1^2)\delta(m_2^2)$ since the final state partons are massless. This contribution is given by

$$\frac{1}{\sigma_0}\left[\frac{d^2\sigma}{dm_1^2 dm_2^2}\right]^{\text{tree + virtual}} = \delta(m_1^2)\delta(m_2^2)$$
$$\times \left\{1 + \frac{\alpha}{4\pi}C_F\left(-\frac{4}{\epsilon^2} - \frac{6}{\epsilon} - 16 + \frac{7\pi^2}{3}\right)\right\} \quad (9)$$

For the real emission process, $e^+e^- \to q\bar{q}g$, there are non trivial contributions to the jet mass distribution.

The differential cross section for real emission of a gluon in $d = 4 - 2\epsilon$ dimensions, leading order in α_s, is given by

$$\frac{1}{\sigma_0}\frac{d^2\sigma}{dudt} = \frac{\alpha_s}{4\pi}C_F \frac{2}{t^{1+\epsilon}u^{1+\epsilon}(1-t-u)^\epsilon}$$
$$\times \left((1-t)^2 + (1-u)^2 - \epsilon(t+u)^2\right).$$

where

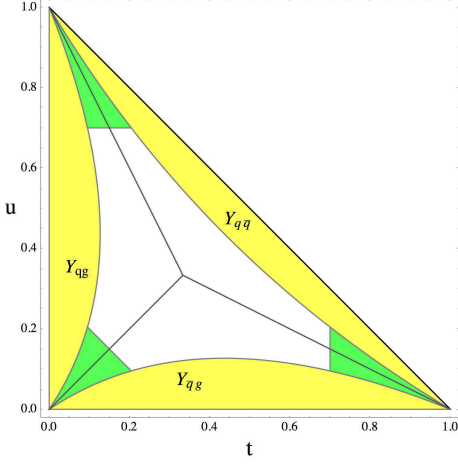$$s = (p_q + p_{\bar{q}}) \quad t = (p_q + p_g) \quad u = (p_{\bar{q}} + p_g). \quad (10)$$

FIG. 3: The allowed kinematic region for $e^+e^- \to q\bar{q}g$ in variables u, t. In the yellow regions, Y_{ij}, the Cambridge/Aachem requirements have been met and partons i, j have been clustered. The green region corresponds to $E_3 < \omega$. In this plot, $R = 0.4$ and $\omega = 0.15Q$.

The kinematically allowed regions are shown in Fig. (3). The yellow regions shown in the figure correspond the regions in phase space where the C/A clustering requirements are met. These regions are labeled Y_{ij} for $i, j = q, \bar{q}, g$, where the subscript corresponds to to the pair of partons that have been clustered. In these regions, the clustered jet has the most energy and so we assign $p_1^\mu = p_i^\mu + p_j^\mu$, whereas p_2^μ is assigned to the momentum of the remaining parton. The shaded green region accounts for the case when the partons are not clustered so there are three massless jets; however, these are still counted as dijet events since $E_3 < \omega$ and thus contribute to the $\delta(m_1^2)\delta(m_2^2)$ part of the distribution. After performing the integrations, the real emission contribution is given by

$$
\frac{1}{\sigma_0}\left[\frac{\mathrm{d}^2\sigma}{\mathrm{d}m_1^2\mathrm{d}m_2^2}\right]^{\text{real}} = \delta(m_1^2)\delta(m_2^2)
$$

$$
+ \frac{\alpha}{4\pi}C_F\delta(m_2^2)\left\{\left(\frac{4}{\epsilon^2} + \frac{6}{\epsilon} + 14 - \frac{5\pi^2}{3}\right.\right.
$$

$$
\left.-8\log\frac{R}{1-R}\log\frac{2\omega}{Q} + f_\omega(R)\right)\delta(m_1^2)
$$

$$
+ \left[\frac{-6 + 8\log\frac{R}{1-R} - 8\log\frac{m_1^2}{Q^2}}{m_1^2}\right]_* + \cdots\right\}
$$

where $\cdots$ denotes terms at higher order in m_1^2 and

$$
f_\omega(R) = -4\log R\log\frac{R}{1-R} - 4\mathrm{Li}_2(1-R) + 4\mathrm{Li}_2(R)
$$

$$
+ \frac{8\omega}{Q}\left(2 + \frac{3}{2}\log\frac{R}{1-R}\right) + \mathcal{O}\left(\frac{\omega^2}{Q^2}\right) \quad (11)
$$

The result of adding the tree, virtual, and real emission contributions is

$$
\frac{1}{\sigma_0}\left[\frac{\mathrm{d}^2\sigma}{\mathrm{d}m_1^2\mathrm{d}m_2^2}\right]_{\text{QCD}} = \delta(m_1^2)\delta(m_2^2) + \frac{\alpha}{4\pi}C_F\delta(m_2^2)
$$

$$
\times\left\{\left(-2 + \frac{2\pi^2}{3} -8\log\frac{R}{1-R}\log\frac{2\omega}{Q} + f_\omega(R)\right)\delta(m_1^2)\right.
$$

$$
\left.+ \left[\frac{-6 + 8\log\frac{R}{1-R} - 8\log\frac{m_1^2}{Q^2}}{m_1^2}\right]_*\right\} + \cdots\right\}. \quad (12)
$$

As we saw in the last section, SCET does not have access to energy ordering of the jets, and the closest observable we can calculate is $\mathrm{d}^2\sigma/\mathrm{d}m_q^2\mathrm{d}m_{\bar{q}}^2$, which we turn to now. The derivation of the factorization theorem for $\mathrm{d}^2\sigma/\mathrm{d}m_q^2\mathrm{d}m_{\bar{q}}^2$ is almost identical to the derivation of the factorization theorem for thrust. For very small $m_{q,\bar{q}}$, the jet size R affects the collinear sector only at subleading power of $m_{q,\bar{q}}$. Therefore, as in the hemisphere mass case, we can use the inclusive (R-independent) jet function. The factorization formula then reads

$$
\frac{1}{\sigma_0}\frac{\mathrm{d}^2\sigma}{\mathrm{d}m_q^2\mathrm{d}m_{\bar{q}}^2} = H(Q^2, \mu)\int \mathrm{d}k_q\mathrm{d}k_{\bar{q}}
$$

$$
\times J(m_q^2 - k_qQ)J(m_{\bar{q}}^2 - k_{\bar{q}}Q)S_R(k_q, k_{\bar{q}}, \omega, \mu). \quad (13)
$$

This soft function is defined as (see [5])

$$
S_R(k_q, k_{\bar{q}}, \omega, \mu) = \sum_{X_s}\langle 0|Y_{\bar{n}}^\dagger Y_n|X_s\rangle\langle X_s|Y_n^\dagger Y_{\bar{n}}|0\rangle
$$

$$
\times \delta(k_q - n\cdot p_X^q)\delta(k_{\bar{q}} - \bar{n}\cdot p_X^{\bar{q}})\theta(\omega - k_0^{\text{out}}), \quad (14)
$$

where n^μ and $\bar{n}^\mu$ are lightlike 4-vectors in the q and $\bar{q}$ jet directions, respectively, Y_n are soft Wilson lines, X_s is the soft radiation, $p_X^q (p_X^{\bar{q}})$ the component of its momenta in the quark (anti-quark) jet, and k_0^{out} is the energy the hardest jet of size R in the inter-jet region.

At order α_s, the soft function can be written as a product of two in-cone soft functions and an out-of-cone soft function: $S_R = S_R^{\text{in}}(k_q, \mu)S_R^{\text{in}}(k_{\bar{q}}, \mu)S_R^{\text{out}}(\omega, \mu)$. The in-cone soft function is the same as for asymmetric thrust and given in Eq. (5). The out-of-cone soft function is

$$
S_R^{\text{out}}(\omega, \mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left[-8\ln\frac{R}{1-R}\ln\frac{2\omega}{\mu}\right.
$$

$$
\left.+ 2\ln^2\frac{R}{1-R} + f_0(R)\right], \quad (15)
$$

with $f_0(R)$ given in Eq.(11). These results agree with limits of expressions given in [13].

Combining this soft function with the hard and inclu-

sive jet functions, we get

$$\frac{1}{\sigma_0}\left[\frac{\mathrm{d}^2\sigma}{\mathrm{d}m_q^2\mathrm{d}m_{\bar{q}}^2}\right]_{\mathrm{SCET}} = \delta(m_q^2)\delta(m_{\bar{q}}^2) + \frac{\alpha_s}{4\pi}C_F\Bigg\{$$

$$\left(-2+\frac{2\pi^2}{3}-8\log\frac{R}{1-R}\log\frac{2\omega}{Q}+f_0(R)\right)\delta(m_q^2)\delta(m_{\bar{q}}^2)$$

$$+\left[\frac{-6+8\log\frac{R}{1-R}-8\log\frac{m_q^2}{Q^2}}{2m_q^2}\right]_*\delta(m_{\bar{q}}^2)$$

$$+\left[\frac{-6+8\log\frac{R}{1-R}-8\log\frac{m_{\bar{q}}^2}{Q^2}}{2m_{\bar{q}}^2}\right]_*\delta(m_q^2)+\cdots\Bigg\} \tag{16}$$

We can now compare SCET directly to the doubly differential distribution of the harder and softer jet mass, which is the natural quantity to measure experimentally, and we have calculated in QCD. The coefficient of $\delta(m_q^2)\delta(m_{\bar{q}}^2)$ in the SCET distribution matches the coefficient of $\delta(m_1^2)\delta(m_2^2)$ from Eq.(12) with $f_0(R)$ appearing instead of $f_\omega(R)$; however, the remaining distribution is symmetric in $m_q \leftrightarrow m_{\bar{q}}$ in SCET, but not symmetric in $m_1 \leftrightarrow m_2$ for QCD. So the doubly differential distributions are different. We anticipated this, but it is still instructive to see that the mass of the hardest jet is not simply related to any projection of the doubly differential distribution in SCET.

Although the hardest jet mass is not reproduced in SCET, one can compute the average of the distributions of the two jet masses. If one integrates one jet mass up to the full kinematic limit, in general there will be non-global logarithms. The result is like comparing the left-hemisphere mass in QCD and in SCET, which do not agree due to non-global logarithms due to extra hard emissions within one jet. However, at small R, there is a natural cutoff $m < Q^2 R$. Then we can compute

$$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}m^2}\right]_{\mathrm{QCD}} = \int_0^{Q^2R}\mathrm{d}m_1^2\int_0^{Q^2R}\mathrm{d}m_2^2\,\frac{\mathrm{d}^2\sigma}{\mathrm{d}m_1^2\mathrm{d}m_2^2}$$

$$\times\frac{1}{2}\left[\delta(m^2-m_1^2)+\delta(m^2-m_2^2)\right]. \tag{17}$$

This can be compared to the analogous expression in SCET, using m_q and $m_{\bar{q}}$ instead of m_1 and m_2. The two distributions agree, as they should.

Rather than look at the average jet mass distribution, we can instead look at a thrust-like variable, the sum of the squares of the jet masses. We call this **jet thrust**, and denote it by τ_ω. By definition, $\tau_\omega = (m_1^2 + m_2^2)/Q^2$ where events with $E_3 < \omega$ have been vetoed. At leading order, $\mathrm{d}\sigma/\mathrm{d}\tau_\omega$ can be calculated by integrating the doubly differential distribution computed above.

$$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega}\right]_{\mathrm{QCD}} = \int_0^{Q^2R}\mathrm{d}m_1^2\int_0^{Q^2R}\mathrm{d}m_2^2\,\frac{\mathrm{d}^2\sigma}{\mathrm{d}m_1^2\mathrm{d}m_2^2}$$

$$\times\delta(\tau_\omega-m_1^2/Q^2-m_2^2/Q^2) \tag{18}$$

The terms singular in τ_ω in full QCD at order α_s, are

$$\frac{1}{\sigma_0}\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega}\right]_{\mathrm{QCD}} = \delta(\tau_\omega)+\frac{C_F\alpha_s}{4\pi}\left[\frac{-8\ln\tau-6+8\ln\frac{R}{1-R}}{\tau_\omega}\right]_+$$

$$+\frac{C_F\alpha_s}{4\pi}\left[-2+\frac{2\pi^2}{3}-8\ln\frac{R}{1-R}\ln\frac{2\omega}{Q}+f_\omega(R)\right]\delta(\tau_\omega). \tag{19}$$

When $\tau_\omega \ll 1$, each jet mass is forced to be small and so SCET is valid in this regime. In the effective theory, we take $\tau_\omega = (m_q^2 + m_{\bar{q}}^2)/Q^2$ and its distribution can be obtained via

$$\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega}\right]_{\mathrm{SCET}} = \int_0^{Q^2R}\mathrm{d}m_q^2\int_0^{Q^2R}\mathrm{d}m_{\bar{q}}^2\,\frac{\mathrm{d}^2\sigma}{\mathrm{d}m_q^2\mathrm{d}m_{\bar{q}}^2}$$

$$\times\delta(\tau_\omega-m_q^2/Q^2-m_{\bar{q}}^2/Q^2) \tag{20}$$

The terms singular in τ_ω in SCET at order α_s, are

$$\frac{1}{\sigma_0}\left[\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega}\right]_{\mathrm{SCET}} = \delta(\tau_\omega)+\frac{C_F\alpha_s}{4\pi}\left[\frac{-8\ln\tau_\omega-6+8\ln\frac{R}{1-R}}{\tau_\omega}\right]_+$$

$$+\frac{C_F\alpha_s}{4\pi}\left[-2+\frac{2\pi^2}{3}-8\ln\frac{R}{1-R}\ln\frac{2\omega}{Q}+f_0(R)\right]\delta(\tau_\omega). \tag{21}$$

This result is the same as the QCD result with $f_0(R)$ appearing instead of $f_\omega(R)$. Thus at LO, SCET reproduces the parts of QCD which are singular in τ_ω, up to power corrections in ω/Q, but including all of the R dependence.

4. PREDICTING HIGHER ORDER TERMS

In the previous sections we explored doubly and singly differential jet mass distributions in SCET and in QCD. For some observables, we saw that SCET reproduces the singular behavior of QCD at order α_s. In particular, for jet thrust, τ_ω, which we defined as the sum of the jet masses squared normalized to the machine energy Q with a energy veto $E < \omega$ on the third hardest jet, we found agreement with QCD for any R at order α_s, up to power corrections in ω/Q. Next, we would like to know to what extent this agreement persists to higher order. The question can be phrased succinctly as asking which of the coefficients C_{ij} in the expansion

$$\tau_\omega\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_\omega} = \sum C_{ij}(R,\omega)\alpha_s^i\ln^j\tau_\omega \tag{22}$$

can be predicted in SCET.

There are two parts to this question. First, if one were able to do calculations to all order in SCET, would the coefficients come out correct? As long as the entire singular region is described by soft and collinear degrees of freedom, the answer should be yes. This is true for τ_ω as
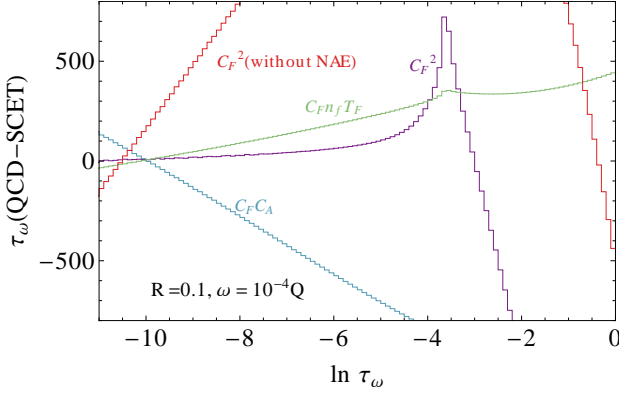
FIG. 4: Difference between QCD and SCET for the $(\alpha_s/4\pi)^2$ coefficient of $\tau_\omega \frac{d\sigma}{d\tau_\omega}$ for the different color structures. After using non-Abelian exponentiation (NAE) the C_F curve should go to zero at small τ_ω, up to power corrections in ω.

long as $\omega \ll Q$ for any R. So SCET should get all the $C_{ij}(R,\omega)$ correct up to power corrections in ω/Q. This can be seen already at order α_s in Eq. (21) where the SCET distribution agrees with the QCD distribution in Eq. (19) exactly in R when terms of order ω/Q or smaller are dropped.

The second part is whether the coefficients $C_{ij}(R,\omega)$ with $i > 1$ can be predicted using factorization. That is, which of these coefficients can we get without actually doing the explicit loop integrals? For example, non-global logarithms appear in the doubly differential hemisphere mass distribution. These are reproduced in SCET [10, 11], but it is so far unclear how to reproduce them without doing calculations to all orders in perturbation theory. We would like to know what can and cannot be predicted, in what limits. We would also like to know to what extent terms which we cannot predict or calculate are actually important.

Since the anomalous dimensions of the jet and hard functions are known to 3-loops, the μ-independence of the τ_ω distribution determines the anomalous dimension of the soft function to 3-loops as well. Note that since the jet and hard functions are completely independent of R and ω, none of the anomalous dimensions have any dependence on R or ω at all. With the finite part of the 1-loop soft function known, this should naively allow us to resum the next-to-next-to-leading logs (NNLL). That would give us all the terms down to $1/\tau_\omega$ in the α_s^2 distribution. This is certainly the case for hemispheres ($R = \frac{1}{2}$). However, for smaller R these anomalous dimensions are not enough to predict the NNLL, or even NLL distribution.

The failure of SCET to describe τ_ω at NLL can be seen already in Eq.(21): at NLL, the 1-loop anomalous dimensions should determine the α_s/τ_ω terms. But since the jet and hard anomalous dimensions lack R dependence, there is no way they can predict the $\alpha_s \ln \frac{R}{1-R}/\tau_\omega$ piece. Thus, without insights into the soft function beyond the original factorization, SCET cannot even resum τ_ω to NLL.

To proceed, we first of all observe that the troublesome term has a coefficient C_F. Due to the non-Abelian exponentiation (NAE) theorems [23, 24] the $\alpha_s^n C_F^n$ terms in the soft function are determined exactly by the 1-loop result. Thus we can resum at least this much R-dependence by simply exponentiating the NLO soft function (in position space). Doing so predicts an NLL term $C_F^2 \alpha_s^2 \ln^2 \tau/\tau$, as well as $\alpha_s^2 \ln \tau_\omega/\tau_\omega$ and α_s^2/τ_ω NNLL terms.

To see how well SCET is doing, we will explore the order α_s^2 distribution of τ_ω. This can be calculated in QCD numerically using the program EVENT 2 [25]. In Figure 4 we show difference between QCD and SCET predictions at order α_s^2 for the different color structures. The C_F^2 curve before and after using non-Abelian exponentiation are shown. We see that after NAE, the C_F^2 curve goes to zero as $\tau_\omega \to 0$, while the $C_F C_A$ and $C_F n_f T_F$ color structures are clearly missing $\ln \tau_\omega Q/\omega$ pieces, which affects the slope of these plots. Note that the curves cross at roughly $\tau_\omega \sim \omega = 10^{-4} \sim e^{-9}$, indicating a missing $\ln \omega/\tau_\omega$ piece. The approach of the C_F term to zero gives us a benchmark for where we should expect the curves to vanish, and how large the ω/Q power corrections are.

A. Refactorization

To further progress, we will now argue that when

$$\omega \lesssim k_L, k_R \qquad \text{and} \qquad R \ll 1 \qquad (23)$$

the soft function factorizes into four parts

$$S_R(k_L, k_R, \omega, \mu) = S_R^{\text{in}}(k_L, \mu) S_R^{\text{in}}(k_R, \mu) S_R^{\text{out}}(\omega, \mu) \\ \otimes S_f(k_L, k_R, \omega), \quad (24)$$

Here, $\otimes$ indicates a convolution. Similar refactorization formulas have appeared before [13, 15] (and after [26]), but of a more general nature. Without specifying the regime in which the refactorization is valid, and more importantly, properties of the objects appearing in the refactorization, there is no content in Eq. (24).

To see that Eq. (24) by itself is contentless, recall that the hemisphere soft function can be written as

$$S_{\text{hemi}}(k_L, k_R, \mu) = \Pi(k_L, \mu)\Pi(k_R, \mu) \otimes S_f(k_L, k_R), \quad (25)$$

where $\Pi(k, \mu)$ are the soft RG factors which compensate for the RG evolution of the jet and hard functions. This form is forced by the hemisphere factorization formula in SCET [5, 6, 27, 28]. But $S_{\text{hemi}} = S_{R=\frac{1}{2}}$, so we can find a trivial solution for (24) by setting $S_R^{in}(k, \mu) = \Pi(k, \mu)$ and $S_R^{\text{out}} = 1$.

We propose that S_R^{in} and S_R^{out} can be thought of as soft functions in their own right, with their own anomalous dimensions. Only S_R^{in} has a $\log \mu$ in its anomalous dimension, since the soft-collinear singularities are within

the jets. We conjecture that the regular anomalous dimensions split as

$$\gamma_S^{\text{out}} = -\Gamma_{\text{cusp}} \ln \frac{R}{1-R} + \gamma_\Delta \qquad (26)$$

$$\gamma_S^{\text{in}} = \Gamma_{\text{cusp}} \ln \frac{R}{1-R} + \gamma_S - \gamma_\Delta$$

where Γ_{cusp} is the cusp anomalous dimension. In particular, all of the R-dependence in the anomalous dimensions in proportional to Γ_{cusp}.

We will argue that $\gamma_\Delta = 0$ as well, which is consistent with the $R = \frac{1}{2}$ limit, although since our refactorization argument is valid at small R, this is not really justified. We also set $S_f(k_L, k_R, \omega) = 1$, which is to say we ignore non-global logs. We will see that the refactorization is still in good agreement with full QCD independent of the non-global log issue. With this conjecture, the refactorization allows us expand around small R to predict non-trivial parts of the τ_ω distribution, for $\omega/Q \lesssim \tau_\omega \ll R \ll 1$, which we will compare our prediction to the fixed order result.

A complete proof of refactorization (or more precisely, why $S_R^{\text{out}}(\omega, \mu) \neq 1$) would ideally involve operator definitions of the components S_R^{in} and S_R^{out}. That way we can ascribe anomalous dimensions to the components, which can, at least in principle, be calculated order-by-order in perturbation theory. However, it may not be possible, or even necessary, to have operator definitions for refactorization. For example, in the hemisphere case, we do not have operator definitions of the components of S_{hemi}. Instead, $\Pi(k, \mu)$ is defined to compensate the hard and jet function evolution, and S_f is defined as whatever is left over. We aim at this point for the more modest goal of having a similar algorithmic definition of the components of S_R, leaving a more complete non-perturbative understanding for future work.

In Section 4 B, we give a heuristic argument for why the soft function should refactorize at small R (and $\omega \lesssim k_L, k_R$). We present phenomenological evidence in support of this refactorization conjecture in Section 4 C.

B. Heuristic argument for refactorization

To justify the refactorization, we will show that the kinematic restriction small R and $\omega \lesssim k_L, k_R$ allows us to apply soft-collinear factorization [29]. The argument we present here is similar to one used in [30] for when two subjects become collinear inside a larger jet.

To begin the argument, let us define the "left" jet to be in the n^μ direction and the "right" jet to be in the $\bar{n}^\mu$ direction. A particle within the left jet is kinematically restricted to have $k_L = k^+ < \frac{R}{1-R} k^-$, where $k^- = \bar{n} \cdot k$ and $k^+ = n \cdot k$. With the on-shell condition, the transverse momentum then scales like $k_\perp \sim \sqrt{k^+ k^-} \sim \sqrt{\frac{R}{1-R}} k^-$. So, when $R \ll 1$, the jet is forced to have collinear scaling $(k^-, k^+, k_\perp) \sim \frac{k_L}{R}(1, R, \sqrt{R})$. Radiation in the right

jet is restricted analogously. Outside the jet, the soft momentum is only required to have $k_{\text{out}} \sim k_L(1, 1, 1)$. Thus, there are three sectors, $|X_{\text{out}}\rangle$, which has ultra-soft scaling and $|X_{\text{in}}^L\rangle$ and $|X_{\text{in}}^R\rangle$ which are soft, with respect to the original collinear momentum, but collinear with respect to the soft radiation outside the jets. We say these have soft-collinear scaling. The relevance of soft-collinear modes was also noticed in [30] in a different context.

It is interesting to observe the soft-collinear modes are formally harder than the original soft modes: in terms of $\tau_\omega \sim k_L/Q$, the soft collinear modes scale like $Q\tau_\omega(\frac{1}{R}, 1, \frac{1}{\sqrt{R}})$ while the original soft modes scaled like $Q\tau_\omega(1, 1, 1)$. However, the soft-collinear modes are still parametrically softer than the original collinear modes, which scale like $Q\tau_\omega(\frac{1}{\tau_\omega}, 1, \frac{1}{\sqrt{\tau_\omega}})$, since $\tau_\omega \ll R$. Thus, both the soft-collinear and ultrasoft modes have eikonal interactions with the original Wilson lines, and hence are present in the original soft function.

Now consider the original Wilson line

$$Y_n(x) = \text{P} \exp \left[ig \int_0^\infty ds \; n \cdot A(x + sn) \right]. \qquad (27)$$

The gauge fields in this Wilson line includes anything with Eikonal interactions with the original collinear fields, which includes both the ultra-soft and soft-collinear modes. Thus, we can write $A_\mu = A_\mu^{ns} + A_\mu^{\bar{n}s} + A_\mu^{\text{us}}$. The soft-collinear modes and ultra-soft modes interact with each other the same way that ordinary collinear and soft modes interact. In any situation in which modes have collinear and ultra-soft scaling, the interactions between the soft and collinear modes can be removed from the Lagrangian through a BPS field redefinition [29]:

$$A_{ns} \to (Y_n^{\text{us}})^\dagger A_{ns} Y_n^{\text{us}}, \qquad A_{\bar{n}s} \to (Y_n^{\text{us}})^\dagger A_{\bar{n}s} Y_n^{\text{us}}. \quad (28)$$

where Y_n^{us} is the same as Y_n but with only the ultra-soft gauge fields involved. After this redefinition, there are no-longer interactions between soft-collinear and ultra-soft modes. Thus, the original Wilson line operator now separates into

$$Y_{\bar{n}}^\dagger Y_n \to (Y_{\bar{n}}^{\text{sc}})^\dagger (Y_{\bar{n}}^{\text{us}})^\dagger (Y_n^{\text{us}})(Y_n^{\text{sc}}) \qquad (29)$$

where Y_n^{sc} is a soft-collinear Wilson line, again given by Eq. (27), but now in terms of decoupled soft-collinear fields. This separation of the soft Wilson line into soft and soft-collinear Wilson lines suggests the refactorized form in Eq. (24).

Note that this derivation is not a complete proof of the refactorization. We have not given operator definitions for S_R^{in} and S_R^{out}, although clearly they should be related to matrix elements of the soft-collinear and ultrasoft Wilson lines respectively. This section should be viewed as an argument for why the soft function might simplify as $R \to 0$. Such simplifications would show up in the calculations of the exact higher-order soft function, which is not yet known. If the reader does not find this argument convincing, he should ignore this section and consider the phenomenological evidence for refactorization at small R which we give below.

C. Predictions from refactorization

To solve the RGE for the soft function, we first extract the anomalous dimensions of the two parts from their NLO expressions. The anomalous dimension of $S_R^{\text{out}}(\omega, \mu)$ has no μ dependence, consistent with there being no soft-collinear radiation in the middle region. The μ dependence of the anomalous dimension of $S_R^{\text{in}}(k, \mu)$ is the same as for the thrust soft function. We can also read off the regular anomalous dimensions at NLO

$$\gamma_S^{\text{out}} = -\Gamma_{\text{cusp}}^{(0)} \ln \frac{R}{1-R} \qquad (30)$$

$$\gamma_S^{\text{in}} = \gamma_S + \Gamma_{\text{cusp}}^{(0)} \ln \frac{R}{1-R}$$

where $\Gamma_{\text{cusp}}^{(0)} = C_F \alpha_s / \pi$ is the one-loop cusp anomalous dimension. Generalizing this to higher orders leads to our conjecture Eq.(26) with $\gamma_\Delta = 0$. As discussed earlier, we are not interested in non-global logs, so we set $S_f(k_L, k_R, \omega) = 1$ for simplicity.

In summary, our conjecture is

$$S_R(k_L, k_R, \omega, \mu) = S_R^{\text{in}}(k_L, \mu) S_R^{\text{in}}(k_R, \mu) S_R^{\text{out}}(\omega, \mu) \quad (31)$$

with $S_R^{\text{in}}(k, \mu)$ having the $\ln(\mu/k)$ part of the anomalous dimension, and the regular anomalous dimensions splitting as $\gamma_S^{\text{out}} = -\Gamma_{\text{cusp}} \ln \frac{R}{1-R}$ and $\gamma_S^{\text{in}} = \gamma_S + \Gamma_{\text{cusp}} \ln \frac{R}{1-R}$.

Although we have not proven the soft anomalous dimension splits up in this way at higher order, we do know that the R dependence must exactly cancel to all orders, which implies it should be universal, naturally suggesting the cusp anomalous dimension. As a check, we also know that as $R \to \frac{1}{2}$, $\gamma_{\text{in}} \to \gamma_S$ and $\gamma_{\text{out}} \to 0$, since this is the hemisphere case. At order α_s^2, this $\boldsymbol{\Gamma}_{\text{cusp}}$ **ansatz** contributes terms like $\Gamma_1 \ln \frac{R}{1-R} \ln \tau_\omega$ to the singular τ_ω distribution, which we confirm exist by numerical comparison to QCD (see below). We do not have a proof for the Γ_{cusp} ansatz or a strong argument about why the regular anomalous dimension cannot split up differently at two loops and beyond.

We also note that the connection between the $\ln R$ terms and the cusp anomalous dimension was pointed out in [13, 15]. In fact, part of the reason we expect that

our Γ_{cusp} ansatz could hold is related to observations about the structure of the divergences in the relevant Feynman diagrams which generate the 1-loop anomalous dimensions, first calculated in these papers. However, without the refactorization at small R, there is no predictive power in this observation. Indeed, generically, there can be an additional anomalous dimension γ_Δ, as indicated in Eq. (26), and γ_Δ can also depend on R. We are arguing that all the singular dependence on R in the anomalous dimension should be given by the cusp anomalous dimension, so that at small R, this γ_Δ is R-independent (and vanishes at one-loop). We cannot rule out such a function which vanishes at $R = \frac{1}{2}$ and goes to a constant starting at two loops. The small R regime in not mentioned at all in [13, 15] (although it is discussed in a paper [31] on non-global logarithms).

As a consequence of refactorization, we can choose separate soft scales for the in-jet and out-of-jet regions, μ_s and μ_ω. Evolving S_R^{out} to μ_ω is straightforward, since there are no scales in its anomalous dimension. Thus

$$S_R^{\text{out}}(\omega, \mu) = \left(\frac{R}{1-R}\right)^{-2A_\Gamma(\mu_\omega, \mu)} S_R^{\text{out}}(\omega, \mu_\omega). \quad (32)$$

The function $A_\Gamma(\nu, \mu)$ is defined in [18]. It follows that choosing $\mu_\omega = \omega$ allows us to resum logs of ω, since there is no other scale in this soft function.

The S_R^{in} soft function is similar to the thrust soft function up to R-dependence. Its evolution equation is similarly solved in Laplace space [18]. The projection relevant for τ_ω adds the k_L and k_R momentum is just the thrust soft function times a new R-dependent factor.

While there may be non-global logs contained in the function $S_f(k_L, k_R, \omega)$ in Eq. (24), for simplicity, we have set $S_f = 1$. Non-global logs are not the subject of this paper. To the extent that our refactorization agrees with the NLO, it is partly because non-global logs are a numerically small effect in the regime we consider. We refer the reader to [26], which appeared after the first version of our paper appeared, and addresses non-global logs for jet thrust and related observables.

Combining the full soft function with the hard and inclusive jet functions gives the resummed τ_ω distribution in SCET

$$\frac{1}{\sigma_0} \left[\frac{d\sigma}{d\tau_\omega}\right]_{\text{SCET}} = \exp\left[4S(\mu_h, \mu_j) + 4S(\mu_s, \mu_j) - 2A_H(\mu_h, \mu_s) + 4A_J(\mu_j, \mu_s)\right] \left(\frac{R}{1-R}\right)^{-2A_\Gamma(\mu_\omega, \mu_s)} \left(\frac{Q^2}{\mu_h^2}\right)^{-2A_\Gamma(\mu_h, \mu_j)}$$

$$\times H(Q^2, \mu_h) S_R^{\text{out}}(\omega, \mu_\omega) \left[\tilde{j}\left(\ln \frac{\mu_s Q}{\mu_j^2} + \partial_\eta, \mu_j\right)\right]^2 \tilde{s}_{\tau_\omega}^{\text{in}}\left(\partial_\eta, \mu_s\right) \frac{1}{\tau_\omega} \left(\frac{\tau_\omega Q}{\mu_s}\right)^\eta \frac{e^{-\gamma_E \eta}}{\Gamma(\eta)}. \quad (33)$$

Note that this expression is explicitly independent of μ. Demanding independence of the various matching

scales fixes the unknown parts of the fixed order soft functions. Both are of the general form of fixed order
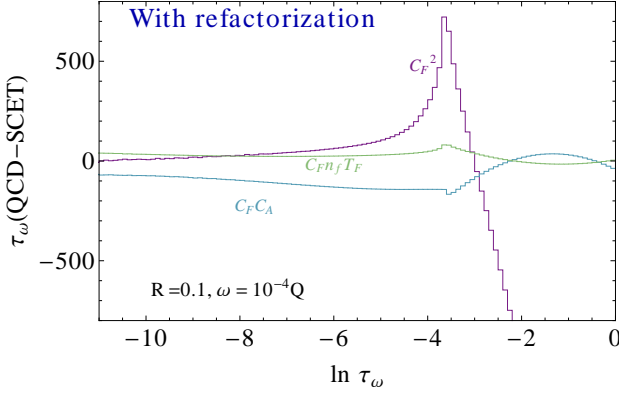
FIG. 5: The difference between coefficient of α_s^2 in $\mathrm{d}\sigma/\mathrm{d}\ln\tau_\omega$ in full QCD and in SCET for $R = 0.1$ and $\omega = 0.0001Q$ after refactorization, but not including the Γ_1 piece.
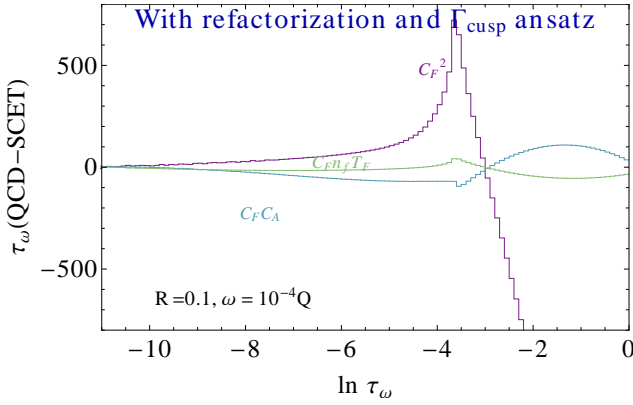

FIG. 6: The difference between coefficient of α_s^2 in $\mathrm{d}\sigma/\mathrm{d}\ln\tau_\omega$ in full QCD and in SCET for $R = 0.1$ and $\omega = 0.0001Q$ after refactorization and including the Γ_1 piece.

expansions in RG-improved perturbation theory, given by $h(L,\mu)$ in Eq. (55) of [18]. In this case $S_R^{\mathrm{out}}(\omega,\mu_\omega) = h(2\ln\frac{2\omega}{\mu_s},\mu_s)$ with all the Γ_{cusp} terms set to zero, since there are no Sudakov logs outside of the jets, and substituting $\gamma_H \to -\gamma_S^{\mathrm{out}}$ with $\gamma_S^{\mathrm{out}} = -\Gamma_{\mathrm{cusp}}\ln\frac{R}{1-R}$ and $c_1^H \to c_1^{\mathrm{out}} = 2C_F\ln^2\frac{R}{1-R} + f_0(R)$. The in-cone soft function has double logs, and is given by $\widetilde{s}_{\tau_\omega}^{\mathrm{in}}(L,\mu) = h(2L,\mu)$ with $\gamma_H \to -\gamma_S^{\mathrm{in}}$, $\gamma_S^{\mathrm{in}} = \gamma_H - 2\gamma_J + \Gamma_{\mathrm{cusp}}\ln\frac{R}{1-R}$, and $c_1^H \to c_1^{\mathrm{in}} = -C_F(\pi^2 + 2\ln^2\frac{R}{1-R})$. Since all the anomalous dimensions are known up to 3-loops, the first unknown terms are the 2-loop constants in the soft functions. Thus, the refactorization formula and the Γ_{cusp} ansatz, that the 2-loop out-of-cone soft function anomalous dimension is proportional to Γ_1, gives us enough for NNLL resummation (again, up to NGLs).

To check our result, we can expand our result to order α_s^2 and compare the resulting distributions to full QCD using EVENT 2. The results should agree up to power corrections in τ_ω, ω/Q and R and up to NGLs. Figure 5 shows the difference between SCET and QCD as a function of $\ln\tau_\omega$ for $R = 0.1$ and $\omega/Q = 0.0001$ using refac-
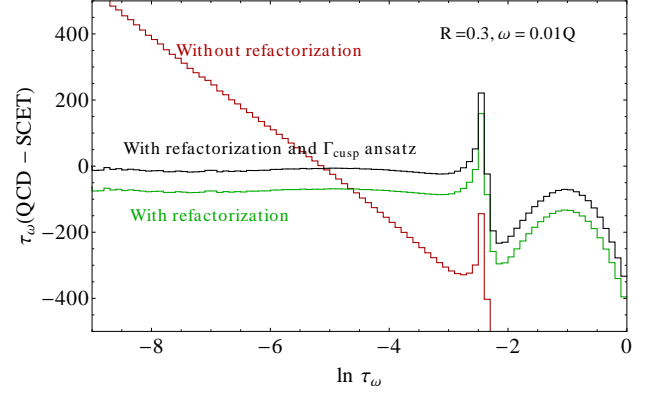

FIG. 7: The difference between coefficient of α_s^2 in $\mathrm{d}\sigma/\mathrm{d}\ln\tau_\omega$ in full QCD and in SCET for $R = 0.3$ and $\omega = 0.01Q$ after refactorization and including the Γ_1 piece.

torization (but setting the 2-loop S^{out} anomalous dimension to 0). We see that now both the $C_A C_F$ and $C_F n_f T_F$ color structures go to zero slope, similar to what was seen already in Figure 4 for the C_F^2 color structure (recall that the C_F^2 part of the 2-loop soft function is known exactly from non-Abelian exponentiation). This indicates that we are getting the $\ln\tau_\omega/\tau_\omega$ terms mostly correct. We then show in Figure 6 the further improvement gained by including the contribution from Γ_1 in the 2-loop soft function anomalous dimension. That the curves now go to zero is a highly non-trivial check. This indicates that we are getting also the $1/\tau_\omega$ terms mostly correct. The refactorization is easiest to confirm in the region with $\tau_\omega \sim \omega$, where the NGLs are necessarily small.

As discussed above, the refactorization we presented is valid only for small $\omega \lesssim \tau_\omega Q$ and small $\tau_\omega \ll R \ll 1$, but we expect it to be relevant even for moderate R. In Figure 7 we show the case with $R = 0.3$. One can see clearly that the refactorization is phenomenologically important even for R which is not terribly small.

5.   CONCLUSIONS

In this paper we have considered a number of important issues which will be relevant to producing accurate distributions of jet masses and other jet substructure observables at colliders. We first considered the possibility of an inclusive jet mass measurement, such as the asymmetric thrust observable. In this case, we were able to resum logarithms of the jet mass to next-to-next-to-leading logarithmic accuracy. These distributions, when expanded to order α_s^2 showed excellent agreement with numerical calculations in full QCD in the threshold region.

At hadron colliders, it will be difficult to use an inclusive jet mass observable. Instead, one must restrict consideration to the hardest jet or the hardest pair of jets in the event. In this case, it is important to veto additional jets, which introduces a veto scale ω. Then there are at least three parameters, ω/Q, where Q is

some hard energy scale, the jet shape τ and the jet size R. The goal in resumming the jet shape would be to predict coefficients of $\alpha_s^i \ln^j \tau$ without doing the full calculation to ith order in perturbation theory. In general, this is difficult, but we argued that there is a refactorization which simplifies the calculation for the observable jet thrust, τ_ω. Jet thrust is defined as the sum of the jet masses squared normalized to the center of mass energy Q, after a vetoing the energy ω of the third hardest jet. We argue that the refactorization holds in the limit $\omega/Q \lesssim \tau_\omega \ll R \ll 1$. We presented kinematic scaling arguments, using the language of Soft-Collinear Effective Theory, to justify the refactorization.

In order to test the refactorization, we compared the predictions to the exact NLO event shape distributions in full QCD. This provided very strong numerical evidence that the refactorization correctly predicts the dominant part of the jet mass distributions. We further argued that the R-dependence may be universal, and therefore may depend on the cusp anomalous dimension. This Γ_{cusp} ansatz was also confirmed by comparison to full QCD. The predictions from refactorization are independent of the separate, and important, issues of non-global logs. NGLs are apparently a numerically small correction to our results, but are not the focus of this paper. Extrapolating away from $R = 0$ our result still provides good agreement with full QCD. Thus the small R expansion seems like a productive direction for further investigation.

The improved agreement with full QCD after refactorization is impressive, but still not completely well understood. It would be helpful to have the full soft function calculated, along the lines of [10, 11], to understand exactly which terms are reproduced and which are not. It would also be helpful to have gauge-invariant operator definitions of the functions into which the soft function factorizes, so that they can be computed directly. Progress along these lines has already appeared, in [30]. Most importantly, it will be critical to construct jet mass observables for hadron colliders which can be computed and resummed to next-to-next-to-leading logarithmic accuracy so that direct comparisons to collider data can be made.

Appendix A: NNLO expansion of SCET results

In this appendix, we give the τ_ω distribution in SCET to order α_s^2. The SCET distribution at order α_s is given in Eq. (21). At order α_s^2, we can write

$$\frac{\tau_\omega}{\sigma_0}\left[\frac{d\sigma}{d\tau_\omega}\right]_{\text{SCET}} = \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{ C_F^2\left[f_0^{C_F} + f_{\text{NAE}}^{C_F}\right] + C_F C_A\left[f_0^{C_A} + f_{\Gamma_1}^{C_A}\right] + C_F n_f T_F\left[f_0^{n_f} + f_{\Gamma_1}^{n_f}\right] + C_F \beta_0 f_{\text{Refact}} \right\} \quad (A1)$$

where $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}n_f T_F$ is the leading β-function coefficient in QCD.

The parts coming from the 1-loop result and the original SCET factorization formula for τ_ω are

$$f_0^{C_F} = 32\ln^3\tau_\omega + \ln^2\tau_\omega\left[72 - 96\ln\frac{R}{1-R}\right]$$
$$+ \ln\tau_\omega\left[32\text{Li}_2(1-R) - 32\text{Li}_2 R + \ln\frac{R}{1-R}\left(64\ln\frac{2\omega}{Q} + 32\ln R - 96\right) - 16\pi^2 + 52\right]$$
$$+ 24\text{Li}_2(1-R) - 24\text{Li}_2(R) + \ln\frac{R}{1-R}\left(48\ln\frac{2\omega}{Q} + 24\ln R + 24\pi^2 - 16\right) + 16\zeta_3 - 8\pi^2 + 9 \quad (A2)$$

$$f_0^{C_A} = 44\ln^2\tau_\omega + \ln\tau_\omega\left[-\frac{352}{3}\ln\frac{R}{1-R} + \frac{8}{3}\pi^2 - \frac{338}{9}\right]$$
$$+ \frac{176}{3}\ln\frac{2\omega}{Q}\ln\frac{R}{1-R} + \frac{88}{3}\text{Li}_2(1-R) - \frac{88}{3}\text{Li}_2 R + \frac{88}{3}\ln\frac{R}{1-R}\ln R + 24\zeta_3 - 57 \quad (A3)$$

$$f_0^{n_f} = -16 \ln^2 \tau_\omega + \ln \tau_\omega \left[\frac{128}{3} \ln \frac{R}{1-R} + \frac{88}{9} \right]$$
$$- \frac{64}{3} \ln \frac{2\omega}{Q} \ln \frac{R}{1-R} - \frac{32}{3} \mathrm{Li}_2(1-R) + \frac{32}{3} \mathrm{Li}_2 R - \frac{32}{3} \ln \frac{R}{1-R} \ln R + 20 \quad \text{(A4)}$$

The part coming from non-Abelian exponentiation is

$$f_{\mathrm{NAE}}^{C_F} = 64 \ln^2 \frac{R}{1-R} \ln \tau_\omega + 32 \ln \frac{R}{1-R} \left(2\mathrm{Li}_2 R - 2 \ln \frac{R}{1-R} \ln \frac{2\omega}{Q} - \ln \frac{R}{1-R} \ln(4R) + \ln(1-R) \ln R - \frac{5\pi^2}{12} \right) \quad \text{(A5)}$$

The part from the soft-refactorization is

$$f_{\mathrm{Refact}} = 16 \ln \frac{\tau_\omega Q}{2\omega} \ln \frac{R}{1-R} + 4\mathrm{Li}_2(R) + \ln^2(1-R) + 2\ln(1-R) \ln R - \ln^2 R - \frac{\pi^2}{3}. \quad \text{(A6)}$$

The parts from the Γ_1 ansatz are

$$f_{\Gamma_1}^{C_A} = \frac{8}{9} \left(67 - 3\pi^2 \right) \ln \frac{R}{1-R}, \qquad f_{\Gamma_1}^{n_f} = -\frac{160}{9} \ln \frac{R}{1-R}. \quad \text{(A7)}$$

Adding these pieces, the complete $C_F C_A$ color structure at order α_s^2 is relatively simple

$$f^{C_A} = 44 \ln^2 \tau_\omega + \left(\frac{8\pi^2}{3} - \frac{176}{3} \ln \frac{R}{1-R} - \frac{338}{9} \right) \ln \tau_\omega + \frac{536}{9} \ln \frac{R}{1-R} + \frac{44}{3} \ln^2 \frac{R}{1-R} - \frac{8\pi^2}{3} \ln \frac{R}{1-R} + 24\zeta_3 - 57 \quad \text{(A8)}$$

Note the ω dependence has completely dropped out of the sum for the $C_F C_A$ (and the $C_F n_f T_F$) color structure order α_s^2. This follows simply from the refactorization. There is ω dependence in the C_F^2 terms.

By explicit calculation, we can also work out the non-global log in the $C_F C_A$ color structure. Rather than use the Cambridge/Aachen algorithm, for simplicity, we calculate the contribution by just including all radiation within R of the jet direction. This produces a non-global log of the form

$$f_{\mathrm{NGL}}^{C_A} = \left(-\frac{16\pi^2}{3} + 32\mathrm{Li}_2 \frac{R^2}{(1-R)^2} \right) \ln \left(\frac{\tau_\omega Q}{2\omega} \right). \quad \text{(A9)}$$

This vanishes at $R = \frac{1}{2}$ and goes to $-\frac{16\pi^2}{3} \ln \frac{\tau_\omega Q}{2\omega}$ at $R = 0$.

[1] J. M. Butterworth, A. R. Davison, M. Rubin and G. P. Salam, Phys. Rev. Lett. **100**, 242001 (2008) [arXiv:0802.2470 [hep-ph]].
[2] D. E. Kaplan, K. Rehermann, M. D. Schwartz and B. Tweedie, Phys. Rev. Lett. **101**, 142001 (2008) [arXiv:0806.0848 [hep-ph]].
[3] A. Abdesselam *et al.*, Eur. Phys. J. C **71**, 1661 (2011) [arXiv:1012.5412 [hep-ph]].
[4] S. Catani, L. Trentadue, G. Turnock and B. R. Webber, Nucl. Phys. B **407**, 3 (1993).
[5] S. Fleming, A. H. Hoang, S. Mantry and I. W. Stewart, Phys. Rev. D **77**, 074010 (2008) [arXiv:hep-ph/0703207].
[6] M. D. Schwartz, Phys. Rev. D **77**, 014026 (2008)

[arXiv:0709.2709 [hep-ph]].
[7] M. Dasgupta and G. P. Salam, Phys. Lett. B **512**, 323 (2001) [arXiv:hep-ph/0104277].
[8] C. W. Bauer and M. D. Schwartz, Phys. Rev. D **76**, 074004 (2007) [arXiv:hep-ph/0607296].
[9] C. W. Bauer and M. D. Schwartz, Phys. Rev. Lett. **97**, 142001 (2006) [arXiv:hep-ph/0604065].
[10] A. Hornig, C. Lee, I. W. Stewart, J. R. Walsh and S. Zuberi, JHEP **1108**, 054 (2011) [arXiv:1105.4628 [hep-ph]].
[11] R. Kelley, M. D. Schwartz, R. M. Schabinger and H. X. Zhu, Phys. Rev. D **84**, 045022 (2011) [arXiv:1105.3676 [hep-ph]].
[12] T. T. Jouttenus, Phys. Rev. D **81**, 094017 (2010)

[arXiv:0912.5509 [hep-ph]].

[13] S. D. Ellis, C. K. Vermilion, J. R. Walsh, A. Hornig and C. Lee, JHEP **1011**, 101 (2010) [arXiv:1001.0014 [hep-ph]].

[14] W. Y. Cheung, M. Luke and S. Zuberi, Phys. Rev. D **80**, 114021 (2009) [arXiv:0910.2479 [hep-ph]].

[15] S. D. Ellis, A. Hornig, C. Lee, C. K. Vermilion and J. R. Walsh, Phys. Lett. B **689**, 82 (2010) [arXiv:0912.0262 [hep-ph]].

[16] G. P. Salam, Eur. Phys. J. C **67**, 637 (2010) [arXiv:0906.1833 [hep-ph]].

[17] R. Kelley and M. D. Schwartz, Phys. Rev. D **83**, 033001 (2011) [arXiv:1008.4355 [hep-ph]].

[18] T. Becher and M. D. Schwartz, JHEP **0807**, 034 (2008) [arXiv:0803.0342 [hep-ph]].

[19] T. Becher, C. Lorentzen and M. D. Schwartz, arXiv:1106.4310 [hep-ph].

[20] A. Banfi, G. P. Salam and G. Zanderighi, Eur. Phys. J. C **47**, 113 (2006) [arXiv:hep-ph/0601139].

[21] J. Gallicchio and M. D. Schwartz, JHEP **1110**, 103 (2011) [arXiv:1104.1175 [hep-ph]].

[22] J. Gallicchio and M. D. Schwartz, Phys. Rev. Lett. **107**, 172001 (2011) [arXiv:1106.3076 [hep-ph]].

[23] J. Frenkel and J. C. Taylor, Nucl. Phys. B **246**, 231 (1984).

[24] J. G. M. Gatheral, Phys. Lett. B **133**, 90 (1983).

[25] S. Catani and M. H. Seymour, Phys. Lett. B **378**, 287 (1996) [arXiv:hep-ph/9602277].

[26] A. Hornig, C. Lee, J. R. Walsh and S. Zuberi, arXiv:1110.0004 [hep-ph].

[27] A. H. Hoang and S. Kluth, arXiv:0806.3852 [hep-ph].

[28] Y. T. Chien and M. D. Schwartz, JHEP **1008**, 058 (2010) [arXiv:1005.1644 [hep-ph]].

[29] C. W. Bauer, D. Pirjol and I. W. Stewart, Phys. Rev. D **65**, 054022 (2002) [arXiv:hep-ph/0109045].

[30] C. W. Bauer, F. J. Tackmann, J. R. Walsh, S. Zuberi,

[31] A. Banfi, M. Dasgupta, K. Khelifa-Kerfa and S. Marzani, JHEP **1008**, 064 (2010) [arXiv:1004.3483 [hep-ph]].

[32] M. Dasgupta and G. P. Salam, JHEP **0203**, 017 (2002) [arXiv:hep-ph/0203009].

[33] After the first version of this paper appeared, the two-loop hemisphere soft function was calculated exactly [10, 11], showing that SCET does indeed reproduce the singular behavior of QCD when $m_L \ll m_R \ll Q$.